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Two Runs Rule based Weighted Alternated Charting Statistic Control Charts to Monitor the Mean Vector of a Bivariate Process

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Abstract: For multivariate processes, while developing the multivariate control charts, the test statistics are the functions of all quality characteristics. In 'Alternated Charting Statistic' (*ACS*) chart, by alternating the charting statistic, only one of the two/three quality characteristics is inspected (measured) per sample. For bivariate processes, in 'Weighted Alternated Charting Statistic' (*WACS*) chart, the weights of the two quality characteristics are considered to decide which one of the two quality characteristics is inspected (measured) per sample. *WACS* chart performs better as compared to the *ACS* chart. In this article, for bivariate processes, two run length based control charts namely, the '*WACS* Synthetic' (*WACS-Syn*) chart and the '*WACS* Group Runs' (*WACS-GR*) chart are proposed. When there is no correlation or small to moderate correlation between the two quality characteristics, it is numerically illustrated that the proposed control charts perform better as compared to the Hotelling χ^2 chart, *ACS* chart and *WACS* chart. Further, *WACS-GR* chart performs significantly better as compared to the *WACS-Syn* chart.

Keywords: Alternated charting statistic, Weighted Alternated charting statistic, *WACS* chart, *ACS-Syn* chart, *ACS-GR* chart.

***MSC 2020 subject classification*:** 62P30

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1. Introduction

In statistical process control, control charts are specially designed tools to detect significant deviation(s) in the process parameters due to assignable causes. For univariate processes, Shewhart $\overline{X}$ chart, *d* chart, *C* chart etc. are widely accepted tools to monitor the mean. Bourke (1991) developed a 'Conforming Run Length' (*CRL*) chart for increases in fraction nonconforming. He defined *CRL* as the number of conforming units between the two successive non conforming units. Wu and Spedding (2000) proposed the 'Synthetic' (*Syn*) chart by mixing the Shewhart $\overline{X}$ chart with the *CRL* chart to detect shifts in the process mean. Wu and Spedding (2000) defined *CRL* as the number of conforming units between the two successive non conforming units including non conforming unit at the end. For the above mentioned charts, 'Average Run Length' (*ARL*) criterion been used. For a univariate case, Wu *et al*. (2001) used the term 'Average Time to Signal' (*ATS*) and used *ATS* criterion in place of *ARL* criterion for the first time. By using *ATS* model, Gadre and Rattihalli (2004) introduced the 'Group Runs' (*GR*) chart by combining the $\bar{X}$ chart and an extended version of the *CRL* chart.

For multivariate processes, Hotelling (1947) developed χ^2 as well as T^2 charts and are frequently used tools to monitor the mean vector. Ghute and Shirke (2008) came up with a variant of the *Syn* chart (*Syn-M*) chart. Similarly, Gadre and Kakade (2016) adapted the Group Runs based control chart of Gadre and Rattihalli (2004) to derive a new chart called *GR-M* chart, which can monitor the process mean vector. It is to be noted that, for multivariate processes, to measure each unit of each quality characteristic is expensive and time consuming (and in some cases this can be destructive too). To reduce time and cost for inspection, Leoni and Costa (2017) developed 'Alternated Charting Statistic' (*ACS*) control chart to monitor bivariate and trivariate processes. For *ACS* chart, two/three quality characteristics (X, Y) / (X, Y, Z) are monitored in an alternating fashion. This chart is operationally easier and efficient as compared to the Hotelling χ^2 chart. For bivariate and trivariate processes, Gadre and Nisha (2021) developed 'ACS Synthetic' (*ACS-Syn*) and 'ACS-Group Runs' (*ACS-GR*) control charts perform better as compared to the *ACS* control chart.

For bivariate processes, Gadre (*) developed a 'Weighted Alternated Charting Statistic' (*WACS*) control chart. In *WACS* chart, the weights of each of the quality characteristics are used to decide how many samples are to be inspected successively corresponding to the respective quality characteristic, and to decide the status of the process. *WACS* chart is efficient as compared to the Hotelling χ^2 chart and the *ACS* chart. *WACS* chart is a Shewhart type control chart. As for the Shewhart type control chart, the runs rule based control charts like, *Syn* chart, *GR* chart etc.; in zero state as well in steady state cases, Shewhart type chart is inferior to the runs rule based control charts. Here two control charts namely, the *WACS-Syn* and the *WACS-GR* are proposed to monitor the mean vector. Numerical illustrations are studied to see the effectiveness of these proposed charts.

This paper is organized as follows. Section-2 covers a brief review of the *Syn-M* and the *GR-M* charts. Some basic notations and operation of the *WACS* chart are given in the same section. Also, the *ATS* expression of the *WACS* chart is given in the same section. Section 3 covers implementation and the design of the *WACS-Syn* and *WACS-GR* charts. In Section-4, numerical illustrations are given to compare the zero-state *ATS* performance of the proposed charts with the existing charts. Also the real life example is considered and the zero state ATS performance of the proposed charts along with the related four charts is carried out in the same section. In Section-5, the 'Steady State *ATS*' (*SSATS*) performance of the charts is studied. Last section covers conclusions of the proposed charts.

2. Literature Review of *Syn-M, GR-M* and *WACS* Charts

In the following, a review of the three existing charts namely the *Syn-M* chart, *GR-M* chart and *WACS* chart are studied. The two proposed charts are based on these three.

2.1. 'Synthetic Chart to Monitor the Mean Vector'(Syn-M) Chart

For multivariate processes, Ghute and Shirke (2008) proposed a *Syn-M* chart, which is a combination of the Hotelling χ^2 chart and the *CRL* chart. They defined *CRL* as the number of conforming samples between the two successive nonconforming samples including nonconforming sample at the end. Let Y_r ($r = 1, 2 \ldots$) be the r^{th} sample (group) based *CRL* and L_s be the control limit of the *Syn-M* chart. At a given sampling point, n items are collected and the test statistic χ^2 of n items is computed. For χ^2 chart, the *UCL* is $UCL_{\chi^2} = \chi^2_{\alpha,p}$, where α is the reciprocal of the in-control *ARL* of χ^2 chart. If the statistic χ^2 is not exceeding UCL_{χ^2}, the sample is considered as a conforming sample; otherwise it is nonconforming. Let L_S be the lower control limit of the *Syn-M* chart. The *Syn-M* chart is said to be out of control if $Y_r \leq L_S$ for the first time. In such a case, a corrective action is taken before continuing the process. If $Yr > L_s$, the process is said to be under control and will not be interrupted. *Syn-M* chart is developed by using 'Average Run Length' (*ARL*) model. *ARL* is the average number of inspected samples (of size n each) by the time the process has gone out of control.

Wu *et al.* (2001) developed a synthetic control chart for increases in fraction non-conforming. They used the *ATS* criterion to obtain the design parameters of this chart. *ATS* is the average number of units inspected (of size n each) by the time the process has gone out of control. If the units produced are per unit of time, *ATS* is n times *ARL*. *ATS* values have two variants. *ATS*0 is the average time to signal in a process when the process has been running smoothly. Such signalling occurs due to random causes and hence indicates false triggers. On the other hand, *ATS*1 is the average time to signal in a process when it has gone out of control. A good charting process is the one which generates a trigger as soon as the process goes out of control. When comparing chart performances, a chart with a larger in control *ATS* (*i.e.*. *ATS*0) indicates a lower false alarm rate compared to other charts. Similarly, a chart with a smaller out of control *ATS* (*i.e. ATS*1) indicates a better detection ability of the process shifts than the other charts. It is the aim of design of experiment to optimize the design parameters so as to minimize *ATS*1, while ensuring that *ATS*0 remains larger than a predefined minimum level. The *ATS* criterion is,

$$\left.\begin{array}{l} \text{Minimize } ATS_1 \\ \text{Subject to} \\ ATS_0 \geq \tau \end{array}\right\} \quad (1)$$

2.2. 'Group Runs Chart to Monitor Mean Vector' (GR-M) Chart

Gadre and Kakade (2016) proposed a *GR-M* chart, which is a combination of Hotelling χ^2 chart and an extended version of *CRL* chart. They also used *ATS* model to develop this chart. Let L_g be the lower control limit of the *GR-M* chart and Y_r (r =*1*, 2, …) be the r^{th} group based *CRL*. If the charting statistic χ^2 is not exceeding UCL_{χ^2}, the sample is considered as a conforming sample; otherwise it is nonconforming. This control chart declares the process as out of control if $Y_l \le L_g$ or for some r (> 1), $Y_r \le L_g$ and $Y_{(r+1)} \le L_g$ for the first time.

2.3. Basic Notations related to WACS chart

For a bivariate process, let $(X, Y)'$ has $N_2(\mu, \Sigma)$ distribution. Let ρ_{xy} be the correlation coefficient between X and Y. Following are some basic notations.

1. $\mu = (\mu_x, \mu_y)'$, $\Sigma = \begin{bmatrix} \sigma_x^2 & \rho\sigma_x\sigma_y \\ \rho\sigma_x\sigma_y & \sigma_y^2 \end{bmatrix}$ are the mean vector and the covariance matrix.
2. $\mu_0 = (\mu_{0x}, \mu_{0y})'$: In control values of process mean vector.
3. $\sigma = (\sigma_x, \sigma_y)'$: The process variability related to the quality characteristic $(X, Y)'$.
4. $\underline{\delta}' = (\delta_x, \delta_y) = \left(\dfrac{\mu_x - \mu_{0x}}{\sigma_{\bar{x}}}, \dfrac{\mu_y - \mu_{0y}}{\sigma_{\bar{y}}}\right)$, where $\sigma_{\bar{x}} = \dfrac{\sigma_x}{\sqrt{2n}}, \sigma_{\bar{y}} = \dfrac{\sigma_y}{\sqrt{2n}}$ are the process variability of $\bar{X}$ and $\bar{Y}$ respectively and it is a shift in the standardized mean vector.
5. $\underline{\delta}'_1 = (\delta_{1x}, \delta_y) = \left(\dfrac{\mu_{1x} - \mu_{0x}}{\sigma_{\bar{x}}}, \dfrac{\mu_{1y} - \mu_{0y}}{\sigma_{\bar{y}}}\right)$: The input parameter.
6. $\underline{D} = (D_x, D_y)'$: The vector of weights of the respective quality characteristics.
7. $D_x = \delta_{1x}$ and $D_y = \delta_{1y}$ are the related weighs.
8. $\underline{\mu}_1 = (\mu_{1x}, \mu_{1y})' = ((\mu_{0x} + (\delta_{1x}\sigma_{\bar{x}})), (\mu_{0y} + (\delta_{1y}\sigma_{\bar{y}})))'$: Out of control values of the process mean vector.
9. $2n$: Sample size.
10. $k_{wa\bar{x}}, k_{wa\bar{y}}$: The coefficients used in the control limits of the *WACS* sub chart
11. $LCL_{wa\bar{x}} = \mu_{0x} - k_{wa\bar{x}}\sigma_{\bar{x}}, UCl_{wa\bar{x}} = \mu_{0x} + k_{wa\bar{x}}\sigma_{\bar{x}}$. Similar notations for $LCL_{wa\bar{y}}$ and $UCL_{wa\bar{y}}$.
12. $ARL_0 = ARL(\underline{0})$: In control *ARL* value; $ATS_0 = ATS(0)$: In control *ATS* value.
13. $ARL_1 = ARL(\underline{\delta}_1)$: Out of control *ARL* value; $ATS_1 = ATS(\underline{\delta}_1)$: Out of control *ATS* value.
14. τ: Minimum required value of ATS_0.
15. p_x, p_y: $P(\delta x), P(\delta_y)$: *P*(getting signal of the *ACS* chart with *X* sample means, given that shift in the standardized mean of *X quality characteristic is* δ_x) and similar meaning for $P(\delta_y)$. Here $p_{x=\phi(-k_x+\delta_x\sqrt{2n})+\phi(-k_x-\delta_x\sqrt{2n})}$ and $p_{y=\phi(-k_y+\delta_y\sqrt{2n})+\phi(-k_y-\delta_y\sqrt{2n})}$. Similarly, $q_x = 1 - p_x$ and $q_y = 1 - p_y$

16. p_{1x}, p_{1y}: $P(\delta_{1x})$, $P(\delta_{1y})$: Power of the *ACS* chart with *X* sample means and *Y* sample means. Here $p_{1x=\phi(-k_x+\delta_{1x}\sqrt{2n})+\phi(-k_x-\delta_{1x}\sqrt{2n})}$ and $p_{1y=\phi(-k_y+\delta_{1y}\sqrt{2n})+\phi(-k_y-\delta_{1y}\sqrt{2n})}$.
17. P_{0x}, p_{0y}: $P(\delta_x = 0)$, $P(\delta_y = 0)$: Probabilities of Type-I error (getting signal under the assumption that the process is running smoothly) of the *ACS* chart with *X* sample means and *Y* sample means respectively. Here $p_{0x} = 2\Phi(-k_x)$ and $p_{0y} = 2\Phi(-k_y)$.

2.4. The Operation of WACS Chart

For a bivariate process, let *X* and *Y* be the two quality characteristics respectively. As mentioned in sub-section 2.3, D_x and D_y are the weights corresponding to the respective variables. There are three possible situations related to the weights. These are $D_x > D_y$, $D_x < D_y$ and $D_x = D_y$. Call these situations as Case-1, Case-2 and Case-3.

Case-1: $D_x > D_y$ is an indicative of *X* observations are moving away from the centre line (μ_{0x}) and reaching close to one of the control limits $(LCL_{wa\bar{x}}, UCL_{wa\bar{x}})$. To have a confirmation about status of the process, it is essential to have one more successive *X* inspection. As $D_x > D_y$, for *Y* inspection no need to carry out two successive *Y* inspections. Stepwise procedure of the operation of *WACS* chart is as follows. Let the term 'Counter' corresponding to the quality characteristics *X* and *Y* be respectively abbreviated as *CNTx* and *CNTy*. Operation of the *WACS* chart for Case-1 is as follows.

Part-I: Start with *X* inspection.

Step-1: Initialize *i* to 0.

Step-2: Initialize *CNTx* to 0.

Step-3: Add *i* by unity. Take i^{th} sample of size $2n$ from $N_2(\mu, \Sigma)$ distribution. Inspect every unit in the i^{th} sample corresponding to *X* inspection only and compute sample mean $\bar{X}$. If $\bar{X} \in (LCL_{wa\bar{x}}, UCL_{wa\bar{x}})$, move to the next step; otherwise go to Step-6.

Step-4: Add *CNTx* by unity. If $CNTx < 2$, go back to Step-3; otherwise go to the next step.

Step-5: Add *i* to unity. Inspect every unit in the i^{th} sample corresponding to *Y* inspection only and compute sample mean $\bar{Y}$ in the i^{th} sample. If $\bar{Y} \in (LCL_{wa\bar{y}}, UCL_{wa\bar{y}})$, go back to Step-2; otherwise go to Step-6.

Step-6: The process has gone out of control. Identify the assignable causes and take a corrective action before restarting the process. Go back to Step-1.

Part-II: Start with *Y* inspection.

Step-1: Initialize *i* to 0.

Step-2: Add *i* by unity. Take i^{th} sample of size $2n$ from $N_2(\mu, \Sigma)$ distribution. Inspect every unit in the i^{th} sample corresponding to *Y* inspection only and compute sample mean

$\overline{Y}$. If $\overline{Y} \in (LCL_{wa\bar{y}}, UCL_{wa\bar{y}})$, initialize *CNTx* to zero and then move to next step; otherwise go to Step-5.

Step-3: Add *i* by unity. Inspect every unit in the *i*th sample corresponding to *X* inspection only and compute $\overline{X}$. If $\overline{Y} \in (LCL_{wa\bar{x}}, UCL_{wa\bar{x}})$, move to the next step; otherwise go to the Step-5.

Step-4: Add *CNTx* by unity. If *CNTx* < 2, go back to Step-3; otherwise go to Step-2.

Step-5: The process has gone out of control. Identify the assignable causes and take a corrective action before restarting the process. Then go back to Step-1.

2.5. Brief Derivation of the ATS Expression of the WACS Chart

For bivariate processes, Gadre (*) derived the *ATS* expressions of *WACS* chart for each of the three cases. A brief of the derivation is given in the Appendix.

For Case-1, ARL_x and ARL_y expressions of the *WACS* chart are

$$ARL_x = \frac{p_x[1+2q_x^2 q_y + q_x(2+q_x^2 q_y) + 3q_x^2 p_y]}{(1-q_x^2 q_y)^2} \quad (2)$$

and

$$ARL_y = \frac{p_y(1+2q_x^2 q_y) + q_y p_x(2+q_x^2 q_y + 3q_x)}{(1-q_x^2 q_y)^2}. \quad (3)$$

Case-2: $D_x < D_y$ is an indicative of *Y* observations are moving away from the centre line (μ_{0y}) and reaching close to the close to one of the control limits $(LCL_{wa\bar{y}}, UCL_{wa\bar{y}})$.

Remark-1: The stepwise procedure of the operation of *WACS* chart of Case-2 is exactly same as that of Case-1 by replacing *X* by *Y* and *Y* by *X* in Part-I and Part-II.

For Case-2, ARL_x and ARL_y expressions of *WACS* chart are

$$ARL_x = \frac{p_y(1+2q_y^2 q_x) + q_x p_y(2+q_y^2 q_x + 3q_y)}{(1-q_y^2 q_x)^2} \quad (4)$$

and

$$ARL_y = \frac{p_y(1+2q_y^2 q_x) + q_y(2+q_y^2 q_x) + 3q_y^2 p_x}{(1-q_y^2 q_x)^2}. \quad (5)$$

Case-3: $D_x = D_y$, the operation of *WACS* chart is exactly same as that of the *ACS* chart. Leoni and Costa [8] derived ARL_x and ARL_y expressions of *ACS* chart. For bivariate processes, the ARL_x and ARL_y expressions of *ACS* chart are,

$$ARL_x = \frac{(p_x(1+q_x q_y) + 2p_y q_x)}{(1-q_x q_y)^2} \quad (6)$$

and

$$ARL_y = \frac{(p_y(1+q_y q_x) + 2p_x q_y)}{(1-q_x q_y)^2}. \quad (7)$$

Note that, $ARL = (ARL_x + ARL_y) / 2$ and $ATS = n(ARL)$.

Remark-2: Note that, for bivariate processes, as the control charts like *ACS* and *WACS* charts, as one of the two quality characteristics is inspected, $ATS = n(ARL)$, though the sample size is $2n$.

3. *WACS-Syn* Chart and *WACS-GR* Chart

Two control charts namely, *WACS-Syn* chart and *WACS-GR* chart are proposed. In both of these charts, *WACS* approach is used to monitor the mean vector so as to detect nonconforming samples.

3.1. Implementation and the Design of WACS-Syn Chart

Wu and Spedding (2000) modified 'Conforming Run Length' (*CRL*) as the number of conforming samples between the two successive nonconforming samples including nonconforming sample at the end. Also, let Yr (r = 1, 2 ...) be the r^{th} *CRL* and L_{was} be the control limit of the *WACS-Syn* chart. Implementation of the *WACS-Syn* chart is described below. In this situation, a sample of size $2n$ is taken from $N_2(\mu, \Sigma)$ distribution. The operation of *WACS-Syn* chart has two levels namely *WACS* based procedure and *Syn* procedure.

WACS based procedure

As mentioned in the sub-section 2.4, if the sample mean of the present quality characteristic in the sample of size $2n$ falls outside the corresponding control limits, call that sample as nonconforming; otherwise it is conforming.

Syn based procedure

As per this procedure, declares the process as out of control if $Y_r \leq L_{was}$ for the first time.

The Design:

Let P_j(j = 0 or 1) be the probability of getting a signal for the *WACS* chart. For j = 0, 1, write,

$$P_j = \frac{1}{ARL_j}. \tag{8}$$

Similar to Wu *et al.* (2001), for j = 0, 1, ATS_j expression of *WACS-Syn* chart is

$$ATS_j = \frac{n}{P_j\{1-(1-P_j)^{L_{was}}\}} \tag{9}$$

As per the *ATS* criterion given in (*1*), using Equation (9), *Lwas* is given by,

$$L_{was} = \frac{\log\left(1-\frac{n}{p_0\tau}\right)}{\log(1-P_0)}. \tag{10}$$

3.2. Implementation and the Design of WACS-GR Chart

The operation of *WACS-GR* chart has two levels namely *WACS* based procedure and *GR* procedure. *WACS* based procedure is as mentioned in *sub-section 3.1*. Let L_{wag} be the control limit of the *WACS-GR* chart.

GR procedure

GR procedure declares the process as out of control, if $Y_1 \leq L_{wag}$ or for some r (> 1), $Y_r \leq L_{wag}$ and $Y_{(r+1)} \leq L_{wag}$ for the first time.

The Design:

Here, the *ATS* model is used to obtain the design parameters for given input parameters. As mentioned in Gadre and Rattihalli [5], the ATS_j expression of the *WACS-GR* chart is,

$$ATS_j = \frac{n}{P_j\{1-(1-P_j)^{L_{wag}}\}^2}. \tag{11}$$

Using *ATS* criterion and from Equation (11), L_{wag} is given by,

$$L_{wag} = \frac{\log\left(1-\sqrt{\dfrac{n}{P_0\tau}}\right)}{\log(1-P_0)}. \tag{12}$$

To compare the performance of the proposed charts with the χ^2 chart, *MV-Syn-M, MV-GR-M*, *ACS*, *ACS-Syn* and *ACS-GR* charts, some numerical illustrations are discussed in the next section.

4. Numerical Illustrations

For the Shewhart type control charts like $\bar{X}$ chart, χ^2 chart and *ACS* chart, when the sample point falls outside / within the control limits, the process is said to be (out of control) / (under statistical control). For such type of control charts, there is no assumption like a head start before monitoring the process.

For run-length based control charts like, *Syn*, *GR* charts, though the sample point falls outside the control limits, it doesn't mean that the process has gone out of control. In such type of control charts, assumptions are made before monitoring the process. These assumptions are called as a head start. Under these assumptions, if the *ATS* performance been studied, it is zero state *ATS* performance.

In the following, zero state *ATS* performance of the χ^2 chart (for two values of ρ), *ACS*, *ACS-Syn*, *ACS-GR* along with *WACS*, *WACS-Syn* and *WACS-GR* charts are studied. Some additional notations are enlisted to be used in Table-2 to Table-5.

Additional Notations

1. n_{wa}, n_{was}, n_{wag} are (1/2) times the sample sizes of the *WACS*, *WACS-Syn* and *WACS-GR* control charts respectively. ATS_{1wa}, ATS_{1was} and ATS_{1wag} are the ATS_1 values of the related control charts. Also, for *ACS, ACS-Syn* and *ACS-GR* charts, n_a, n_{as}, n_{ag} and ATS_{1a}, ATS_{1as} and ATS_{1ag} are used.
2. n_{ch}, and ATS_{1ch} are the sample size and ATS_1 of the χ^2 chart. Also for *MV-Syn-M* and *MV-GR-M* charts, n_{ms}, n_{mg} and ATS_{1ms} and ATS_{1mg} are used.
3. Similar notations are used for the coefficient of the control limit '*k*' for the six charts.

4.1. Example-1:

Here, the input parameters are $\underline{\delta}_1 = (0,\ 0.5)'$ and $\tau = 370$. The design parameters for the related six charts along with respective *ATSI* values are as follows.

χ^2 chart (When ρ = 0.7): $n_{ch} = 15$, $k_{ch} = 6.42$, $ATS_{1ch} = 23.2314$

χ^2 chart (When ρ = 0.5): $n_{ch} = 19$, $k_{ch} = 5.94$, $ATS_{1ch} = 30.9906$

***ACS* chart:** $n_a = 12$, $k_{ax} = 3.29$, $k_{ay} = 1.86$, $ATS_{1a} = 27.2111$

***ACS-Syn* chart:** *nas* = 8, *kasx* = 2.96, *kasy* = 1.5, *Lmas* = 5, A*TS1mas* = 20.4394

***WACS* chart:** $n_{wa} = 11$, $k_{wax} = 3.12$, $k_{way} = 2.02$, $ATS_{1wa} = 24.7793$

***WACS-Syn* chart:** $n_{was} = 7$, $k_{wasx} = 3.70$, $k_{wasy} = 1.66$, $L_{was} = 5$, $ATS_{1was} = 18.2308$

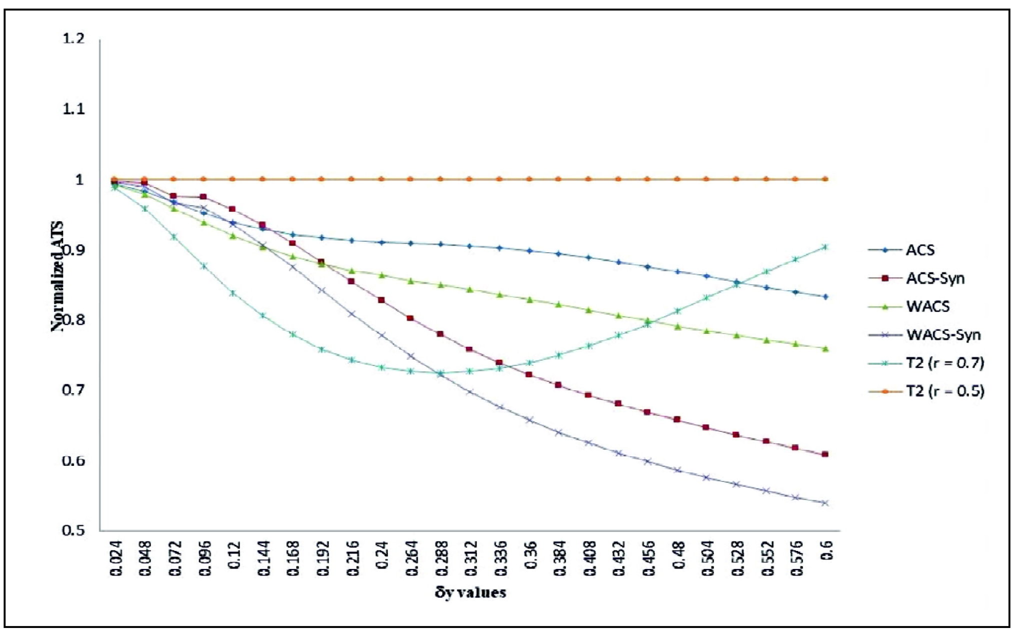

Figure 1: Graph of Normalized ATSs of Six Charts

Table 1: Comparative Study of the Normalized *ATS* values of the Six Charts

Sr. No.	*(δx, δy)*	*ACS*	*ACS-Syn*	*WACS*	*WACS-Syn*	χ^2 *(ρ = 0.7)*	χ^2 *(ρ = 0.5)*
1	(0, 0.024)	**0.9938**	**0.9976**	**0.9931**	**0.9963**	**0.9982**	**1**
2	(0, 0.048)	**0.9826**	**0.9947**	**0.9785**	**0.9802**	**0.9832**	**1**
3	(0, 0.072	**0.9678**	**0.9768**	**0.9589**	**0.9672**	**0.9617**	**1**
4	(0,0.096	**0.9527**	**0.9757**	**0.9384**	**0.9606**	**0.9372**	**1**
5	(0, 0.12)	**0.9396**	**0.9580**	**0.9196**	**0.9364**	**0.9123**	**1**
6	(0, 0.144)	**0.9294**	**0.9354**	**0.9037**	**0.9074**	**0.8886**	**1**
7	(0, 0.168)	**0.9220**	**0.9095**	**0.8906**	**0.8755**	**0.8668**	**1**
8	(0, 0.192)	**0.9170**	**0.8821**	**0.8800**	**0.8423**	**0.8473**	**1**
9	(0, 0.216)	**0.9136**	**0.8545**	**0.8711**	**0.8096**	**0.8300**	**1**
10	(0, 0.24)	**0.9113**	**0.8278**	**0.8635**	**0.7784**	**0.8149**	**1**
11	(0, 0.264),	**0.9094**	**0.8026**	**0.8566**	**0.7493**	**0.8017**	**1**
12	(0, 0.288)	**0.9076**	**0.7795**	**0.8501**	**0.7227**	**0.7903**	**1**
13	(0, 0.312)	**0.9053**	**0.7584**	**0.8433**	**0.6986**	**0.7806**	**1**
14	(0, 0.336)	**0.9025**	**0.7394**	**0.8365**	**0.6771**	**0.7725**	**1**
15	(0, 0.36)	**0.8989**	**0.7223**	**0.8294**	**0.6578**	**0.7657**	**1**
16	(0, 0.384)	**0.8944**	**0.7069**	**0.8221**	**0.6405**	**0.7603**	**1**
17	(0, 0.408)	**0.8892**	**0.6929**	**0.8147**	**0.6251**	**0.7561**	**1**
18	(0, 0.432)	**0.8832**	**0.6801**	**0.8071**	**0.6113**	**0.7531**	**1**
19	(0, 0.456)	**0.8766**	**0.6683**	**0.7996**	**0.5988**	**0.7510**	**1**
20	(0, 0.48)	**0.8695**	**0.6572**	**0.7922**	**0.5873**	**0.7499**	**1**
21	(0, 0.504)	**0.8621**	**0.6468**	**0.7850**	**0.5767**	**0.7496**	**1**
22	(0, 0.528)	**0.8545**	**0.6368**	**0.77822**	**0.5668**	**0.7501**	**1**
23	(0, 0.552)	**0.8470**	**0.6272**	**0.7717**	**0.5575**	**0.7512**	**1**
24	(0, 0.576)	**0.8396**	**0.6179**	**0.7657**	**0.5487**	**0.7529**	**1**
25	(0, 0.6)	**0.8326**	**0.6089**	**0.7601**	**0.5404**	**0.7550**	**1**

Note-1: Related to six control charts, *ATS* (Green) ≤ *ATS* (Blue) ≤ *ATS* (Red) ≤ *ATS* (Violet) ≤ *ATS* (Brown) ≤ *ATS* (Black)

Result

1. The *ATS* comparison is studied through Table-1 and Figure-1, keeping δ_x as zero, for various values of δ_y between 0.024 and 0.6. From Figure-1, it is observed that, for $\delta_y >$ 0.168, *WACS-Syn* chart signals faster than the remaining five charts.

4.2. Example-2:

In this example, for the comparison purpose, as mentioned in Leoni and Costa [8], Gadre and Nisha (2021), Gadre and Kakade (2016) and Gadre (10), for τ = 370, 15 combinations of δ_1 are considered. Considering all 15 combinations of the input parameters ($\underline{\delta}_1$, τ), values of the design parameters along with respective *ATS*1 values are computed for two control charts namely, *WACS* and *WACS-Syn* charts; and are given in Table 2.

For four control charts namely, *Syn-M* (for ρ = 0.7 and for ρ = 0.5), *ACS-Syn* and *WACS-Syn*, values of the design parameters along with respective ATS_1 are given in Table 3. Also, for *WACS-Syn* and *WACS-GR* charts, values of the design parameters along with respective ATS_1 are computed and are given in Table 4. Table-5 is useful to compare the performance of *GR-M* (for ρ = 0.7 and for ρ = 0.5), *ACS-GR* and *WACS-GR* charts.

Table 2: ATS_1 Comparison of *WACS* and *WACS-Syn* control charts

$\delta_1 = (\delta_{1x}, \delta_{1y})$	*WACS*				*WACS-Syn*				
	n	k_x	k_y	ATS_1	n	k_x	k_y	L_g	ATS_1
(0, 0.50)	**11**	3.12	2.02	**24.7793**	**7**	3.70	1.66	5	**18.2308**
(0, 0.75)	**6**	3.35	2.26	**13.4207**	**4**	3.26	1.80	5	**9.5472**
(0, 1)	**4**	3.52	2.41	**8.5541**	**2**	3.10	2.00	6	**6.1048**
(0, 1.50)	**2**	3.9	2.65	**4.4442**	**1**	3.56	2.14	6	**3.1164**
(0.50, 0.50)	**12**	2.12	2.16	**19.3067**	**8**	1.65	1.76	3	**13.7590**
(0.50, 0.75)	**7**	2.41	2.32	**12.5512**	**4**	2.01	1.89	4	**8.7765**
(0.50, 1)	**4**	2.97	2.44	**8.4432**	**3**	2.21	1.91	4	**5.8818**
(0.50, 1.50)	**2**	3.99	2.65	**4.4402**	**1**	3.56	2.14	6	**3.1114**
(0.75, 0.75)	**7**	2.31	2.39	**10.3563**	**4**	1.82	1.92	3	**7.1319**
(0.75, 1)	**5**	2.42	2.5	**7.6807**	**3**	1.88	1.96	3	**5.1734**
(0.75, 1.50)	**2**	3.45	2.66	**4.4124**	**1**	3.09	2.15	6	**3.0911**
(1, 1)	**4**	2.52	2.58	**6.5605**	**3**	1.88	1.99	3	**4.4307**
(1, 1.50)	**2**	3	2.71	**4.2310**	**2**	1.88	2.11	3	**2.8698**
(1.50, 1.50)	**2**	2.74	2.83	**3.4173**	**1**	2.19	2.25	4	**2.3704**

Note-2: Related to the control charts, ATS_1 (Green) ≤ ATS_1 (Red) and n(Green) ≤ n(Red)

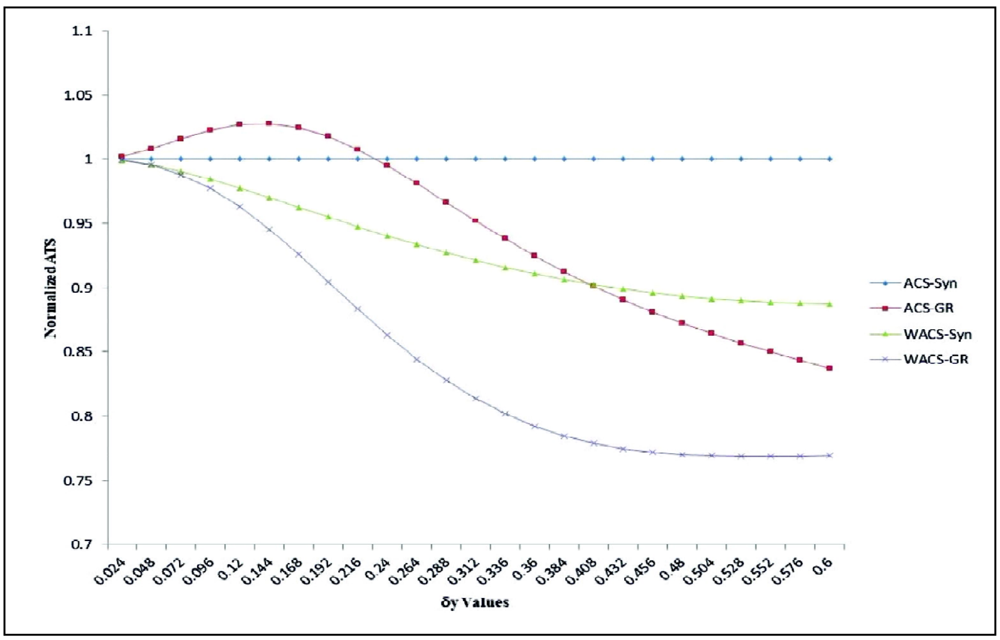

Figure 2: Graph of Normalized ATSs of Four Charts

Table 3: ATS_1 Comparison of the *ACS-Syn* and *WACS-Syn* control charts

$\underline{\delta}_1 = (\delta_{1x}, \delta_{1y})$	*MV-Syn-M*								*ACS-Syn*					*WACS-Syn*				
	ρ = 0.7				ρ = 0.5													
	n	k	L	ATS_1	n	k	L	ATS_1	n	k_x	k_y	L_s	ATS_1	n	k_x	k_y	L_g	ATS_1
(0, 0.50)	**10**	4.61	3	**17.0646**	**14**	4.26	3	**23.2041**	**8**	2.96	1.50	5	**20.4394**	**7**	3.70	1.66	5	**18.2308**
(0, 0.75)	**6**	5.15	3	**8.8504**	**7**	4.99	3	**22.1267**	**4**	2.84	1.73	6	**10.7752**	**4**	3.26	1.80	5	**9.5472**
(0, 1)	**3**	5.86	3	**5.5094**	**5**	5.34	3	**7.5327**	**3**	2.87	1.76	5	**6.7238**	**2**	3.10	2.00	6	**6.1048**
(0, 1.50)	**2**	6.28	3	**2.7391**	**3**	5.48	2	**3.8870**	**2**	3.00	1.80	5	**3.6267**	**1**	3.56	2.14	6	**3.1164**
(0.50, 0.50)	**15**	4.18	3	**25.2432**	**14**	4.26	3	**23.2041**	**8**	1.65	1.76	3	**13.7590**	**8**	1.65	1.76	3	**13.7590**
(0.50, 0.75)	**9**	4.72	3	**15.2477**	**9**	4.72	3	**14.8278**	**5**	2.01	1.77	4	**9.3481**	**4**	2.01	1.89	4	**8.7765**
(0.50, 1)	**6**	5.15	3	**8.9998**	**6**	5.15	3	**9.5493**	**3**	2.41	1.83	5	**6.4177**	**3**	2.21	1.91	4	**5.8818**
(0.50, 1.50)	**3**	5.86	3	**4.0287**	**3**	5.86	3	**4.6809**	**2**	2.69	1.88	5	**3.5826**	**1**	3.56	2.14	6	**3.1114**
(0.75, 0.75)	**8**	4.85	3	**13.4043**	**7**	4.99	3	**12.1267**	**4**	1.82	1.92	3	**7.1319**	**4**	1.82	1.92	3	**7.1319**
(0.75, 1)	**6**	5.15	3	**9.5114**	**6**	5.15	3	**8.9693**	**3**	1.99	1.88	3	**5.4513**	**3**	1.88	1.96	3	**5.1734**
(0.75, 1.50)	**3**	5.86	3	**4.5303**	**3**	5.86	3	**4.8317**	**2**	2.20	1.98	4	**3.3769**	**1**	3.09	2.15	6	**3.0911**
(1, 1)	**5**	5.34	3	**8.3541**	**5**	5.34	3	**7.5317**	**3**	1.88	1.99	3	**4.4307**	**3**	1.88	1.99	3	**4.4307**
(1, 1.50)	**3**	5.86	3	**4.8224**	**3**	5.86	3	**4.6809**	**2**	2.02	2.02	3	**3.0062**	**2**	1.88	2.11	3	**2.8698**
(1.5, 1.5)	**3**	5.86	3	**4.2350**	**3**	5.48	2	**3.8870**	**1**	2.19	2.25	4	**2.3704**	**1**	2.19	2.25	4	**2.3704**

*Note 3:*Related to the four control charts this Note is similar to note-2.

Results

2. From Table-2, for all combinations of $\underline{\delta}_1$, $ATS_{1was} \leq ATS_{1wa}$ and $n_{was} \leq n_{wa}$.
3. From Table-3, for all combinations of $\underline{\delta}_1$, $ATS_{1was} \leq ATS_{1as}$ and $n_{was} \leq n_{as}$. For $\rho < 0.7$, for all combinations of $\underline{\delta}_1$, $ATS_{1was} \leq ATS_{1chs}$ and $n_{was} \leq n_{chs}$ except for $\underline{\delta}_1 = \underline{0}$. Also for $\rho = 0.7$, for $\delta_{1x} > 0$, $ATS_{1was} \leq ATS_{1chs}$ and $n_{was} \leq n_{chs}$ except for $\underline{\delta}_1 = \underline{0}$.
4. From Table-4, for all combinations of $\underline{\delta}_1$, $ATS_{1wag} \leq ATS_{1was}$ and $n_{wag} \leq n_{was}$.

Table 4: ATS_1 Comparison of *WACS-Syn* and *WACS-GR* control charts

$\underline{\delta}_1 = (\delta_{1x}, \delta_{1y})$	*WACS-Syn*					*WACS-GR*				
	n	k_x	k_y	L_g	ATS_1	n	k_x	k_y	L_g	ATS_1
(0, 0.50)	**7**	3.70	1.66	5	**18.2308**	**6**	2.86	1.52	6	**15.7255**
(0, 0.75)	**4**	3.26	1.80	5	**9.5472**	**3**	2.93	1.65	6	**8.0569**
(0, 1)	**2**	3.10	2.00	6	**6.1048**	**2**	3.01	1.72	6	**4.9551**
(0, 1.50)	**1**	3.56	2.14	6	**3.1164**	**1**	3.47	1.83	6	**2.4587**
(0.50, 0.50)	**8**	1.65	1.76	3	**13.7590**	**7**	1.42	1.53	3	**11.6380**
(0.50, 0.75)	**4**	2.01	1.89	4	**8.7765**	**4**	1.61	1.69	4	**7.2989**
(0.50, 1)	**3**	2.21	1.91	4	**5.8818**	**2**	2.20	1.73	5	**4.8267**
(0.50, 1.50)	**1**	3.56	2.14	6	**3.1114**	**1**	3.47	1.83	6	**2.4556**
(0.75, 0.75)	**4**	1.82	1.92	3	**7.1319**	**3**	1.66	1.77	4	**5.9456**
(0.75, 1)	**3**	1.88	1.96	3	**5.1734**	**2**	1.80	1.77	4	**4.2843**
(0.75, 1.50)	**1**	3.09	2.15	6	**3.0911**	**1**	2.42	1.88	6	**2.4068**
(1, 1)	**3**	1.88	1.99	3	**4.4307**	**2**	1.77	1.79	4	**3.6132**
(1, 1.50)	**2**	1.88	2.11	3	**2.8698**	**1**	2.05	1.91	5	**2.2508**
(1.50, 1.50)	**1**	2.19	2.25	4	**2.3704**	**1**	1.84	1.95	4	**1.7960**

Note 4: As note-2.

5. From Table-5, for all combinations of $\underline{\delta}_1$, $ATS_{1wag} \leq ATS_{1ag}$ and $n_{wag} \leq n_{ag}$. For $\rho < 0.7$, for all combinations of $\underline{\delta}_1$, $ATS_{1wag} \leq ATS_{1chg}$ and $n_{wag} \leq n_{chg}$ except for $\underline{\delta}_1 = \underline{0}$. Also for $\rho = 0.7$, for $\delta_{1x} > 0$, $ATS_{1wag} \leq ATS_{1chg}$ and $n_{wag} \leq n_{chg}$ except for $\underline{\delta}_1 = \underline{0}$.

Result:

6. The *ATS* comparison is studied through Table 6 and Figure 2, keeping δ_x as zero, for various values of δ_y between 0.024 and 0.6. From Figure 2, it is observed that, for all 25 values of δ_y, *WACS-GR* chart performs better as compared to *ACS-Syn* and *ACS-GR* charts. For $\delta_y > 0.048$, *WACS-GR* chart also performs as compared to *WACS-Syn* chart.
7. The *ATS* comparison is studied through Table 7 and Figure 3, keeping δ_x as zero, for various values of δ_y between 0.024 and 0.6. From Figure 3, it is observed that, for $\delta_y > 0.12$, *WACS-GR* chart performs better as compared to *WACS* and *WACS-Syn* charts.

In the following, a real life example of the bivariate process is considered to see performance of the proposed control charts as compared to the related control charts.

4.3. A Real Life Example

The data are from most important part, calliper of the brake system that measured the Lug-hole CD which is distance from two bottom holes of the calliper (X) with the specification 142.05 ± 0.75 mm and diameter which is the distance of centre hole (Y) with the specification of 51.07 ± 0.15 mm for 800 observations. According to historical

Table 5: ***ATS1*** **Comparison of** ***ACS-GR*****and** ***WACS-GR*** **control charts**

$\underline{\delta}_1 = (\delta_{1x}, \delta_{1y})$	*MV-GR-M*								*ACS-GR*					*WACS-GR*				
	$\rho = 0.7$				$\rho = 0.5$													
	n	*k*	*L*	ATS_1	*n*	*k*	*L*	ATS_1	*n*	k_x	k_y	L_s	ATS_1	*n*	k_x	k_y	L_g	ATS_1
(0, 0.50)	**9**	3.74	3	**14.6049**	**11**	3.58	3	**19.9627**	**6**	2.99	1.38	6	**17.6929**	6	2.86	1.52	6	**15.7255**
(0, 0.75)	**5**	4.17	3	**7.4466**	**6**	4.04	3	**10.2586**	**3**	2.91	1.52	6	**9.1731**	3	2.93	1.65	6	**8.0569**
(0, 1)	**3**	4.54	3	**4.5325**	**4**	4.33	3	**6.2971**	**2**	2.96	1.64	7	**5.6067**	2	3.01	1.72	6	**4.9551**
(0, 1.50)	**2**	4.33	2	**2.4052**	**2**	4.86	3	**3.1272**	**1**	3.00	1.76	7	**2.8017**	1	3.47	1.83	6	**2.4587**
(0.50, 0.50)	**13**	3.46	3	**22.0807**	**11**	3.58	3	**19.9627**	**7**	1.42	1.53	3	**11.6380**	7	1.42	1.53	3	**11.6380**
(0.50, 0.75)	**8**	3.82	3	**12.9871**	**8**	3.82	3	**12.6446**	**4**	1.73	1.6	4	**7.7934**	4	1.61	1.69	4	**7.2989**
(0.50, 1)	**5**	4.17	3	**7.5734**	**5**	4.17	3	**8.0420**	**2**	2.4	1.65	6	**5.3873**	2	2.20	1.73	5	**4.8267**
(0.50, 1.50)	**2**	4.83	3	**3.2815**	**2**	5.18	4	**3.9834**	**1**	3.00	1.76	7	**2.7758**	1	3.47	1.83	6	**2.4556**
(0.75, 0.75)	**7**	3.92	3	**11.3681**	**6**	4.04	3	**10.2586**	**3**	1.66	1.77	4	**5.9456**	3	1.66	1.77	4	**5.9456**
(0.75, 1)	**5**	4.17	3	**8.0096**	**5**	4.17	3	**7.5474**	**3**	1.68	1.58	3	**4.5575**	2	1.80	1.77	4	**4.2843**
(0.75, 1.50)	**2**	5.18	4	**3.8260**	**3**	4.54	3	**4.1006**	**1**	2.53	1.76	6	**2.6991**	1	2.42	1.88	6	**2.4068**
(1, 1)	**4**	4.33	3	**7.0315**	**4**	4.33	3	**6.2971**	**2**	1.77	1.79	4	**3.6132**	2	1.77	1.79	4	**3.6132**
(1, 1.50)	**7**	4.54	3	**4.0947**	**2**	5.18	4	**3.9834**	**1**	2.2	1.79	5	**2.4630**	1	2.05	1.91	5	**2.2508**
(1.50, 1.50)	**2**	4.83	3	**3.5158**	**2**	4.83	3	**3.1272**	**1**	1.84	1.95	4	**1.7960**	1	1.84	1.95	4	**1.7960**

Note 5: As per note-3

information about this type of Calliper, the in-control mean vector and covariance matrix were taken as $\underline{\mu}_0 = \begin{bmatrix} 142.5 \\ 51.7 \end{bmatrix}$. Here $\Sigma 0$ is taken as $\Sigma 0 = \begin{bmatrix} 1 & .5 \\ 0.5 & 1 \end{bmatrix}$. Assuming that the in-control process has a $N_2(\underline{\mu}_0, \Sigma 0)$ distribution, the process is stable with respect to its mean vector. As per the specifications, $\underline{\mu}_1$ is taken as $\underline{\mu}_1 = \begin{bmatrix} 142.8 \\ 51.22 \end{bmatrix}$. Now $\underline{\delta}_1 = (\delta_{1x}, \delta_{1y})'$ is computed by

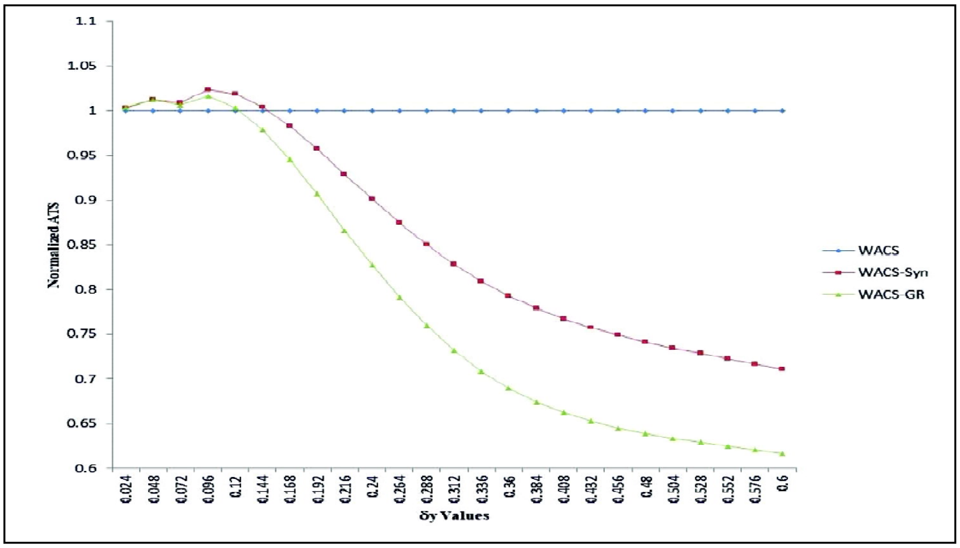

Figure 3: Graph of Normalized ATSs of Three Charts

Table 6: Comparative Study of the Normalized *ATS* values of the Four Charts

Sr. No.	*(δ_x, δ_y)*	*ACS-Syn*	*ACS-GR*	*WACS-Syn*	*WACS-GR*
1	(0, 0.024)	**1**	**1.002297**	**0.998653**	**0.999546**
2	(0, 0.048)	**1**	**1.007835**	**0.995502**	**0.995612**
3	(0, 0.072	**1**	**1.015771**	**0.990166**	**0.987703**
4	(0,0.096	**1**	**1.022082**	**0.984498**	**0.977655**
5	(0, 0.12)	**1**	**1.026686**	**0.977535**	**0.963173**
6	(0, 0.144)	**1**	**1.027677**	**0.970116**	**0.945516**
7	(0, 0.168)	**1**	**1.024578**	**0.962521**	**0.925603**
8	(0, 0.192)	**1**	**1.017612**	**0.954948**	**0.90455**
9	(0, 0.216)	**1**	**1.007455**	**0.947535**	**0.883456**
10	(0, 0.24)	**1**	**0.994988**	**0.940382**	**0.863254**
11	(00.264),	**1**	**0.981105**	**0.933561**	**0.84464**
12	(0, 0.288)	**1**	**0.966593**	**0.927132**	**0.828069**
13	(0, 0.312)	**1**	**0.952083**	**0.921149**	**0.813784**
14	(0, 0.336)	**1**	**0.938045**	**0.915661**	**0.801857**
15	(0, 0.36)	**1**	**0.92472**	**0.910649**	**0.792165**
16	(0, 0.384)	**1**	**0.912336**	**0.906185**	**0.784588**
17	(0, 0.408)	**1**	**0.900941**	**0.902256**	**0.77888**
18	(0, 0.432)	**1**	**0.890524**	**0.898866**	**0.774766**
19	(0, 0.456)	**1**	**0.881015**	**0.895986**	**0.771971**
20	(0, 0.48)	**1**	**0.872327**	**0.893595**	**0.770221**
21	(0, 0.504)	**1**	**0.864347**	**0.891655**	**0.769255**
22	(0, 0.528)	**1**	**0.85695**	**0.890117**	**0.768842**
23	(0, 0.552)	**1**	**0.850024**	**0.888934**	**0.768793**
24	(0, 0.576)	**1**	**0.84348**	**0.88806**	**0.768939**
25	(0, 0.6)	**1**	**0.837214**	**0.887423**	**0.769158**

Note 6: As per Note-1.

Table 7: Comparative Study of the Normalized *ATS* values of the Three Charts

Sr. No.	(δ_x, δ_y)	*WACS*	*WACS-Syn*	*WACS-GR*
1	(0, 0.024)	**1**	**1.003211**	**1.004108**
2	(0, 0.048)	**1**	**1.011969**	**1.012081**
3	(0, 0.072	**1**	**1.008683**	**1.006175**
4	(0,0.096	**1**	**1.023669**	**1.016554**
5	(0, 0.12)	**1**	**1.018298**	**1.003337**
6	(0, 0.144)	**1**	**1.004178**	**0.978715**
7	(0, 0.168)	**1**	**0.982996**	**0.945292**
8	(0, 0.192)	**1**	**0.957248**	**0.906728**
9	(0, 0.216)	**1**	**0.929395**	**0.866543**
10	(0, 0.24)	**1**	**0.901435**	**0.827501**
11	(0.264),	**1**	**0.874781**	**0.791459**
12	(0, 0.288)	**1**	**0.850176**	**0.759335**
13	(0, 0.312)	**1**	**0.828478**	**0.731914**
14	(0, 0.336)	**1**	**0.809413**	**0.708814**
15	(0, 0.36)	**1**	**0.79304**	**0.689858**
16	(0, 0.384)	**1**	**0.779133**	**0.674584**
17	(0, 0.408)	**1**	**0.76737**	**0.662438**
18	(0, 0.432)	**1**	**0.757405**	**0.652836**
19	(0, 0.456)	**1**	**0.748857**	**0.645206**
20	(0, 0.48)	**1**	**0.741377**	**0.639019**
21	(0, 0.504)	**1**	**0.73465**	**0.633803**
22	(0, 0.528)	**1**	**0.728404**	**0.629162**
23	(0, 0.552)	**1**	**0.722445**	**0.624805**
24	(0, 0.576)	**1**	**0.716629**	**0.620503**
25	(0, 0.6)	**1**	**0.710875**	**0.616139**

Note 7: Related to the three control charts this note is similar to Note-1.

using the relationship between $\underline{\mu}_0$, $\underline{\mu}_1$ and $\underline{\delta}_1$. Here $\underline{\delta}_1 = (2.3717, 0.4743)'$ and $\rho = 0.5$ is considered.

ATS comparison of the proposed charts with the related four charts is carried out. Let $\tau = 370$. By using *ATS* criterion, following are the design parameters of the seven related control charts along with the ATS_1 values.

1. **χ^2 Chart:** $n_{ch} = 4$, $k_{ch} = 9.055$, $ATS_{1ch} = 5.65$
2. ***ACS* Chart:** $n_a = 1$, $k_{ax} = 3.84$, $k_{ay} = 2.79$, $ATS_{1a} = 2.3012$
3. ***ACS-Syn* chart:** $n_{as} = 1$, $k_{asx} = 2.01$, $k_{asy} = 3$, $L_{as} = 5$, $ATS_{1as} = 1.7086$
4. ***ACS-GR* chart:** $n_{ag} = 1$, $k_{agx} = 1.75$, $k_{agy} = 2.31$, $L_{ag} = 5$, $ATS_{1ag} = 1.6043$
5. ***WACS* Chart:** $n_{wa} = 1$, $k_{wax} = 3.8$, $k_{way} = 2.88$, $ATS_{1wa} = 2.0771$
6. ***WACS-Syn* chart:** $n_{was} = 1$, $k_{wasx} = 2.82$, $k_{wasy} = 2.13$, $L_{was} = 5$, $ATS_{1was} = 1.6411$

Table 8: *SSATS* performance of the proposed charts with the *ACS*, ACS-Syn and ACS-GR charts

δ_x	δ_y	*ACS*	*ACS-Syn*	*WACS-Syn*	*ACS-GR*	*WACS-GR*				
		SSATS	*SSATS*	*Adj. SSATS*	*SSATS*	*Adj. SSATS*	*SSATS*	*Adj. SSATS*	*SSATS*	*Adj. SSATS*
0	0	**370.5700**	424.0581	**370.5700**	420.9041	**370.57**	471.3278	**370.5698**	471.5664	**370.57**
0	0.50	**27.4688**	26.9033	**23.5099**	24.0171	**21.1451**	26.9815	**21.21354**	23.7017	**18.62546**
0	0.75	**18.8796**	16.7406	**14.6290**	14.8534	**13.07715**	14.8518	**11.67686**	14.0585	**11.04756**
0	1	**18.0072**	15.1401	**13.2304**	13.3362	**11.74138**	12.4349	**9.77663**	12.3124	**9.675427**
0	1.50	**17.9769**	14.9898	**13.0991**	13.1315	**11.56116**	12.0265	**9.45554**	12.0185	**9.444472**
0.50	0.50	**21.9052**	22.9489	**20.0543**	23.4272	**20.62564**	23.6259	**18.5753**	21.2962	**16.73515**
0.50	0.75	**16.6456**	15.4245	**13.4789**	14.6342	**12.88416**	13.9397	**10.9597**	13.1833	**10.3598**
0.50	1	**16.0425**	14.1263	**12.3445**	13.1556	**11.58238**	11.8448	**9.3123**	11.6181	**9.129826**
0.50	1.50	**16.0213**	14.0025	**12.2363**	12.9552	**11.40594**	11.4844	**9.0293**	11.3498	**8.918989**
0.75	0.75	**13.6243**	12.8621	**11.2397**	13.6545	**12.02162**	11.9377	**9.3857**	11.4134	**8.968967**
0.75	1	**13.1100**	12.0712	**10.5486**	12.3446	**10.86836**	10.4844	**8.2431**	10.1939	**8.01065**
0.75	1.50	**13.2987**	11.9935	**10.4807**	12.1637	**10.7091**	10.2242	**8.0385**	9.9766	**7.83989**
1	1	**12.1416**	10.1809	**8.8967**	10.6324	**9.360917**	8.7354	**6.8680**	8.5298	**6.702954**
1	1.50	**12.1336**	10.1364	**8.8579**	10.4913	**9.236691**	8.5813	**6.7468**	8.3692	**6.57675**
1.50	1.50	**12.0000**	9.3397	**8.1616**	8.2973	**7.305062**	7.1778	**5.6434**	7.1577	**5.62472**

7. ***WACS-GR* chart:** $n_{wag} = 1$, $k_{wax} = 1.98$, $k_{way} = 1.9$, $L_{wag} = 5$, $ATS_{1wag} = 1.5519$

The results observed for the seven control charts are as per expectation. In the next section, meaning of steady state and its *ATS* performance of some of the related charts are studied.

5. Steady State *ATS* Performances of the two Charts

As the Markov chain representation of the run-length based control charts like *WACS-Syn* chart and *WACS-GR* chart, such charts have more than one non-absorbing states, the future behaviour of the chart can be studied by using 'Steady State *ATS*' (*SSATS*). The *SSATS* measures average time to signal, when the effect of head start has been faded away.

5.1. Comparison of the 'Steady State ATS' (SSATS) Performance of the ACS, ACS-Syn and ACS-GR Charts

It is to be noted that for Shewhart type control charts, as there is only one non absorbing states, zero state *ATS* and *SSATS* are same. In order to compare the steady state performance of the *WACS-Syn* and *WACS-GR* charts along with *ACS-Syn* and *ACS-GR* charts, it is necessary to determine the *SSATS*. A Mat-Lab code has been developed to do this. To illustrate for a bivariate process, consider the input parameters $\delta_{1x} = 0$ and $\delta_{1y} = 0.5$. The design parameters of the four charts are given Table 3.

For any run length based control chart, the *SSATS* should be larger than zero state *ATS*. But if the signal depends on one point only, both the *ATS* values will be same. Hence the performance of the two charts can be compared by making the *SSATS0* of the two charts same. So, the 'Adjusted *SSATS*' ((*Adj*. $SSATS_j$)) of chart II with respect to chart I is given by,

$$[Adj.\ SSATS\ (\underline{\delta})]_{II} = \{[SSATS\ (\underline{\delta})]'_{II}\ [(SSATS)_0)]_{II}\}\{[(SSATS)_0]_I\} \tag{13}$$

In the computation of *Adj*. *SSATS*, *ACS* control chart is considered as chart-I. The *SSATS* values for *ACS-Syn*, *ACS-GR*, *WACS-Syn* and *WACS-GR* charts are given in the following table. Note that *Adj*. *SSATS* of *ACS* chart is same as *SSATS* of that chart.

From Table-8, we have the following results for all combinations of $\underline{\delta}$.

Results

6. *Adj*. *SSATS* of *WACS-GR* chart is less as compared to the remaining four charts.
7. *Adj*. *SSATS* of *WACS-Syn* chart is less as compared to the *ACS-Syn* chart except for $\underline{\delta} \neq \underline{0}$, $\delta_x = 0$ and for $\underline{\delta} = (0.5, 0.5)'$.

8. Conclusions

In this article, the runs rules are used to propose two new control charts namely 'Weighted Alternated Charting Statistic Synthetic' (*WACS-Syn*) chart and 'Weighted Alternated Charting Statistic Group Runs' (*WACS-GR*) chart to control the mean vector of a bivariate normal process. An interesting problem is for p (> 3) to develop 'Multivariate WACS' (*MWACS*) as well the related run length based control charts for a multivariate normal process.

Also, it is possible to develop 'Multi-Attribute WACS' (*MA-WACS*) control chart for $P > 3$, where, P is the number of variables / attributes.

In zero state, for all combinations of the input parameters, *WACS-GR* chart performs better as compared to the *WACS, ACS-Syn WACS-Syn, ACS-GR,* charts; and *WACS-Syn* chart is better as compared to the *WACS, ACS* and *ACS-Syn* chart. Also the sample sizes of *WACS-GR* chart are least among related charts.

If the input parameters are not known, one may think of developing the *ACS* as well as the *WACS* based control charts for estimated parameters.

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Appendix

As mentioned in sub-section 2.4, we consider three cases, namely, $D_x > D_y$, $D_x < D_y$ and $D_x = D_y$.

Case-1: Derivation of ATS *of* WACS *Control Chart*

Suppose the data are drawn from N_2 ($\underline{\mu}$, Σ) distribution. Let X and Y be the first and the second quality characteristic respectively. As mentioned in sub-section 2.3, p_x and p_y are probabilities of a unit being defective corresponding to the X and Y respectively. Now, '*ARL* of the first quality characteristic (X, Say)' (ARL_x) is,

$$ARLs = p_x\left\{\sum_{i=1}^{\infty}(3i-2)(q_x^2q_y)^{(i-1)} + q_x\sum_{i=1}^{\infty}(3i-1)(q_x^2q_y)^{(i-1)}\right\} + 3(q_x^2p_y)\sum_{i=1}^{\infty}i(q_x^2q_y)^{(i-1)}$$

$$= \frac{p_x[1+2q_x^2q_y+q_x(2+q_x^2q_y)+3q_x^2p_y]}{(1+q_x^2q_y)^2} \qquad \text{(A.1)}$$

and, that of the second quality characteristic (Y, Say) is,

$$ARLs = p_y\left\{\sum_{i=1}^{\infty}(3i-2)(q_x^2q_y)^{(i-1)} + q_yp_x\sum_{i=1}^{\infty}(3i-1)(q_x^2q_y)^{(i-1)}\right\} + 3q_yq_xp_x\sum_{i=1}^{\infty}i(q_x^2q_y)^{(i-1)}$$

$$= \frac{p_y(1+2q_x^2q_y+q_yp_x(2+q_x^2q_y)+3q_x)}{(1+q_x^2q_y)^2}. \qquad \text{(A.2)}$$

As $ATS = n\,(ARL)$, $ATS = \dfrac{n(ARL_x + ARL_y)}{2}$

Remark-1:

Derivation of the *ATS* expression of the *WACS* chart for Case-2 is exactly same as that of Case-1 by replacing X by Y and Y by X in Part-I and Part-II.

For Case-2, ARL_x and ARL_y expressions of *WACS* chart are

$$ARL_x = \frac{p_x(1+2q_y^2q_x)+q_xp_y(2+q_y^2q_x+3q_y)}{(1+q_y^2q_x)^2} \qquad \text{(A.3)}$$

and

$$ARL_y = \frac{p_y(1+2q_y^2q_x)+q_y(2+q_y^2q_x)+3q_y^2p_x}{(1+q_y^2q_x)^2} \qquad \text{(A.4)}$$

Remark-3

For Case-3, *ARLx* and *ARLy* expressions are same as given in Leoni and Costa (2017). These are, $ARL_x = \frac{(p_x(1+q_x q_y)+2p_y q_x)}{(1-q_x q_y)^2}$ and $ARL_y = \frac{(p_y(1+q_y q_x)+2p_x q_y)}{(1-q_x q_y)^2}$.